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A Bivariate Gamma Distribution Whose Marginals are Finite Mixtures of Gamma Distributions

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Abstract
In this article a new bivariate distribution, whose both the marginals are finite mixtures of gamma distributions, has been defined. Several of its properties such moments, correlation coefficients, measure of skewness, moment generating function, Rényi and Shannon entropies have been derived. Simulation study has been conducted to evaluate the performance of maximum likelihood method.

Keywords Bivariate Distribution, Beta Distribution, Entropy, Information Matrix, Gamma Distribution, Simulation

AMS 2010 subject classifications 62H10, 62E15

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1. Introduction

The univariate gamma distribution is one of the most commonly used statistical distributions to analyze skewed data in many disciplines and has been studied extensively in scientific literature. The chi-square distribution, which is of utmost importance in statistical inference, is a special case of gamma distribution. Probability distributions such as exponential and Erlang are also special cases of the gamma distribution. Several univariate generalizations and variants of gamma distribution have also been developed and applied in various areas.

The univariate gamma distribution has been generalized to the bivariate case in many different ways and many forms of bivariate gamma distribution are available. Several techniques to generate bivariate distributions have also been proposed in the scientific literature, *e.g.*, see Balakrishnan and Lai [1], Mardia [12], and Zhang and Singh [30].

Bivariate gamma distributions have found useful applications in many areas. They have been used for representing joint probabilistic properties of multivariate hydrological events such as floods and storms or in the modeling of rainfall at two nearby rain gauges, data obtained from rainmaking experiments, the dependence between annual streamflow and aerial precipitation, wind gust modeling (Smith and Adelfang [25], Smith, Adelfang, and Tubbs [26]), and the dependence between rainfall and runoff (see Nadarajah and Gupta [16], Nadarajah [14, 15] and references therein). For an interesting review of bivariate gamma distributions for hydrological application, the reader is referred to Yue, Ouarda, and Bobée [29] and Zhang and Singh [30].

Nadarajah [13] has listed a number of bivariate gamma distributions such as McKay's bivariate gamma distribution, Dussauchoy and Berland's bivariate gamma distribution, Cherians bivariate gamma distribution, Arnold and Strauss' bivariate gamma distribution, Becker and Roux's bivariate gamma distribution, and Smith and Adelfang's bivariate gamma distribution.

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Chatelain and Tourneret [3] proposed a family of bivariate gamma distributions whose marginals have different shape parameters and indicated its usefulness in detecting changes in two synthetic radar aperture (SAR) images acquired by different sensors and having different numbers of looks. Nadarajah [14] defined gamma-exponential distribution whose margins have the gamma and the exponential distributions. Nadarajah [15], by using two independent gamma variables, constructed a bivariate distribution which has gamma and beta distributions as its marginals. By using conditional approach (see Section 5.6 of Balakrishnan and Lai [1], Nagar, Nadarajah and Okorie [19]), Nagar, Zarrazola and Sánchez [18] constructed a bivariate distribution whose marginal laws are gamma and Macdonald. Piboongungon, Aalo, Iskander and Efthymoglou [20] derived the bivariate correlated generalised gamma fading distribution and have indicated its use in radar signal processing and communications. The bivariate gamma distribution has also been defined as the joint distribution Z_1^2 and Z_2^2, where both Z_1 and Z_2 are standard normal variables with the correlation coefficient ρ (Vere-Jones [27], Maejima and Ueda [11]). Saboor and Ahemad [23] introduced a bivariate gamma-type density function of two variables involving a confluent hypergeometric function. Bondesson [2] reviewed some results for generalized gamma convolutions and derived new bivariate gamma distributions from shot-noise models.

For a review of known bivariate distributions, we refer the readers to Mardia [12], Kotz, Balakrishnan and Johnson [8], and Balakrishnan and Lai [1]. For an excellent review on univariate and bivariate gamma distributions the reader is referred to Saboor, Provost and Ahmad [22]. For matrix variate generalization of the gamma distribution one can consult Gupta and Nagar [6].

In this paper, we introduce a bivariate gamma distribution whose marginals are finite mixtures of gamma distributions and study its properties. This is the first bivariate distribution of its kind and is suitable for bivariate data with negative correlation. We organize our article as follows. In Section 2 we propose the bivariate gamma distribution and discuss some of its properties. Sections 3 and 4 deal with several results such as moments, correlation coefficients, measure of skewness, moment generating function, etc. Entropies such as Rényi and Shannon are derived in Section 5. Distributions of sum, quotient and product and many other distributional results are obtained in Section 6. Estimation of parameters and Fisher information matrix are discussed in Section 7. Finally, in the last section, simulation study is conducted to evaluate the performance of maximum likelihood method.

2. The bivariate gamma distribution

The random variables X_1 and X_2 are said to have a bivariate gamma distribution with parameters α, β and k, denoted by $(X_1, X_2) \sim \text{BGa}(\alpha, \beta, k)$, if their joint density is given by

$$f(x_1, x_2; \alpha, \beta, k) = C(\alpha, \beta, k)(x_1 x_2)^{\alpha-1}(x_1 + x_2)^k \exp\left[-\frac{1}{\beta}(x_1 + x_2)\right], \tag{1}$$

where $x_1 > 0$, $x_2 > 0$, $\alpha > 0$, $\beta > 0$, $k \in \mathbb{N}_0$ and $C(\alpha, \beta, k)$ is the normalizing constant.

By integrating the joint density of X_1 and X_2 over its support set, the normalizing constant is derived as

$$[C(\alpha, \beta, k)]^{-1} = \int_0^\infty \int_0^\infty (x_1 x_2)^{\alpha-1}(x_1 + x_2)^k \exp\left[-\frac{1}{\beta}(x_1 + x_2)\right] dx_1\, dx_2. \tag{2}$$

Now, expanding $(x_1 + x_2)^k$ using binomial theorem and integrating x_1 and x_2, we obtain

$$\begin{aligned}[C(\alpha, \beta, k)]^{-1} &= \sum_{j=0}^{k} \binom{k}{j} \int_0^\infty \int_0^\infty x_1^{\alpha+j-1} x_2^{\alpha+k-j-1} \exp\left[-\frac{1}{\beta}(x_1 + x_2)\right] dx_1\, dx_2 \\ &= \beta^{2\alpha+k}\Gamma^2(\alpha) \sum_{j=0}^{k} \binom{k}{j} (\alpha)_j\, (\alpha)_{k-j}\, .\end{aligned}$$

Finally, using Lemma A.1, we get

$$[C(\alpha,\beta,k)]^{-1}=\beta^{2\alpha+k}\Gamma^{2}\left(\alpha\right)\left(2\alpha\right)_{k} \tag{3}$$

and

$$C(\alpha,\beta,k)=\frac{\Gamma\left(2\alpha\right)}{\beta^{2\alpha+k}\Gamma^{2}\left(\alpha\right)\Gamma\left(2\alpha+k\right)}. \tag{4}$$

An alternative way to compute (2) is to substitute $s=x_1+x_2$ and $r=x_1/(x_1+x_2)$ and integrate s and r by using gamma and beta integrals. Since this approach works for all $k>0$, we will use it to compute Shannon entropy.

Let us now briefly discuss the shape of (1). The first order derivatives of $\ln f(x_1,x_2;\alpha,\beta,k)$ with respect to x_1 and x_2 are

$$f_{x_1}(x_1,x_2)=\frac{\partial\ln f(x_1,x_2;\alpha,\beta,k)}{\partial x_1}=\frac{\alpha-1}{x_1}+\frac{k}{x_1+x_2}-\frac{1}{\beta} \tag{5}$$

and

$$f_{x_2}(x_1,x_2)=\frac{\partial\ln f(x_1,x_2;\alpha,\beta,k)}{\partial x_2}=\frac{\alpha-1}{x_2}+\frac{k}{x_1+x_2}-\frac{1}{\beta}, \tag{6}$$

respectively. Setting (5) and (6) to zero, the only stationary point of (1) is obtained as

$$a=x_{10}=x_{20}=\frac{\beta(2\alpha+k-2)}{2},$$

where $2\alpha+k-2>0$. Computing second order derivatives of $\ln f(x_1,x_2;\alpha,\beta,k)$, from (5) and (6), we get

$$f_{x_1x_1}(x_1,x_2)=\frac{\partial^2\ln f(x_1,x_2;\alpha,\beta,k)}{\partial x_1^2}=-\frac{\alpha-1}{x_1^2}-\frac{k}{(x_1+x_2)^2}, \tag{7}$$

$$f_{x_1x_2}(x_1,x_2)=\frac{\partial^2\ln f(x_1,x_2;\alpha,\beta,k)}{\partial x_1\partial x_2}=-\frac{k}{(x_1+x_2)^2}, \tag{8}$$

and

$$f_{x_2x_2}(x_1,x_2)=\frac{\partial^2\ln f(x_1,x_2;\alpha,\beta,k)}{\partial x_2^2}=-\frac{\alpha-1}{x_2^2}-\frac{k}{(x_1+x_2)^2}. \tag{9}$$

Further, from (7), (8) and (9), we get

$$f_{x_1x_1}(a,a)=-\frac{4\alpha+k-4}{(2\alpha+k-2)^2\beta^2},$$

$$f_{x_1x_2}(a,a)=-\frac{k}{(2\alpha+k-2)^2\beta^2},$$

$$f_{x_2x_2}(a,a)=-\frac{4\alpha+k-4}{(2\alpha+k-2)^2\beta^2}$$

and finally

$$f_{x_1x_1}(a,a)f_{x_2x_2}(a,a)-[f_{x_1x_2}(a,a)]^2=\frac{8(\alpha-1)}{(2\alpha+k-2)^3\beta^4}.$$

Now, observe that

- If $\alpha > 1$, then $f_{x_1x_1}(a,a)f_{x_2x_2}(a,a) - [f_{x_1x_2}(a,a)]^2 > 0$, $f_{x_1x_1}(a,a) < 0$ and $f_{x_2x_2}(a,a) < 0$ and therefore (a,a) is a maximum point.
- If $0 < \alpha < 1$ and $2\alpha + k - 2 > 0$, then $f_{x_1x_1}(a,a)f_{x_2x_2}(a,a) - [f_{x_1x_2}(a,a)]^2 < 0$, and therefore (a,a) is a saddle point.

Figure 1 illustrates the shape of the pdf (1) for selected values of α and β and k.

It can easily be observed that (X_1, X_2) and (X_2, X_1) are identically distributed and hence X_1 and X_2 are exchangeable.

A distribution is said to be negatively likelihood ratio dependent if the density $f(x_1, x_2)$ satisfies

$$f(x_1,x_2)f(x_1^*,x_2^*) \le f(x_1,x_2^*)f(x_1^*,x_2)$$

for all $x_1 > x_1^*$ and $x_2 > x_2^*$ (Lehmann [9], Tong [28]). One can check that the bivariate distribution defined by the density (1) is negatively likelihood ratio dependent.

By integrating x_2 in (1) the marginal density of X_1 is obtained as

$$f_{X_1}(x_1) = C(\alpha,\beta,k)\int_0^\infty (x_1x_2)^{\alpha-1}(x_1+x_2)^k \exp\left[-\frac{1}{\beta}(x_1+x_2)\right] dx_2. \tag{10}$$

Substituting x_2/x_1 = z in (10), the marginal density of X_1 is rewritten as

$$f_{X_1}(x_1) = C(\alpha,\beta,k)x_1^{2\alpha+k-1}\exp\left(-\frac{x_1}{\beta}\right)\int_0^\infty z^{\alpha-1}(1+z)^k \exp\left(-\frac{x_1 z}{\beta}\right) dz. \tag{11}$$

Now, Writing $(1+z)^k$ using binomial theorem and integrating z in (11), the marginal density of X_1 is derived as

$$\begin{aligned} f_{X_1}(x_1) &= C(\alpha,\beta,k)x_1^{\alpha+k-1}\beta^\alpha \exp\left(-\frac{x_1}{\beta}\right)\sum_{j=0}^k \binom{k}{j}\Gamma(\alpha+j)\left(\frac{x_1}{\beta}\right)^{-j} \\ &= C(\alpha,\beta,k)x_1^{\alpha-1}\beta^{\alpha+k} \exp\left(-\frac{x_1}{\beta}\right)\sum_{j=0}^k \binom{k}{j}\Gamma(\alpha+k-j)\left(\frac{x_1}{\beta}\right)^{j}. \end{aligned} \tag{12}$$

Likewise, the marginal density of X_2 is obtained as

$$f_{X_2}(x_2) = C(\alpha,\beta,k)x_2^{\alpha+k-1}\beta^\alpha \exp\left(-\frac{x_2}{\beta}\right)\sum_{j=0}^k \binom{k}{j}\Gamma(\alpha+j)\left(\frac{x_2}{\beta}\right)^{-j}. \tag{13}$$

Thus, the marginal density of X_i is a finite mixture of gamma densities. Figure 2 shows some plots of the marginal density of X_1 for $\beta = 2, k = 0, 1, \ldots, 20$ and some values of α. Substituting $u = z/(1+z)$ with $dz = (1-u)^{-2}du$ in (11), one gets

$$\begin{aligned} f_{X_1}(x_1) &= C(\alpha,\beta,k)x_1^{2\alpha+k-1}\exp\left(-\frac{x_1}{\beta}\right) \\ &\quad\times \int_0^1 u^{\alpha-1}(1-u)^{-(\alpha+k+1)}\exp\left[-\frac{x_1 u}{\beta(1-u)}\right] du. \end{aligned} \tag{14}$$

Now, writing

$$(1-u)^{-(\alpha+k+1)}\exp\left[-\frac{x_1 u}{\beta(1-u)}\right] = \sum_{j=0}^\infty u^j L_j^{(\alpha+k)}\left(\frac{x_1}{\beta}\right)$$

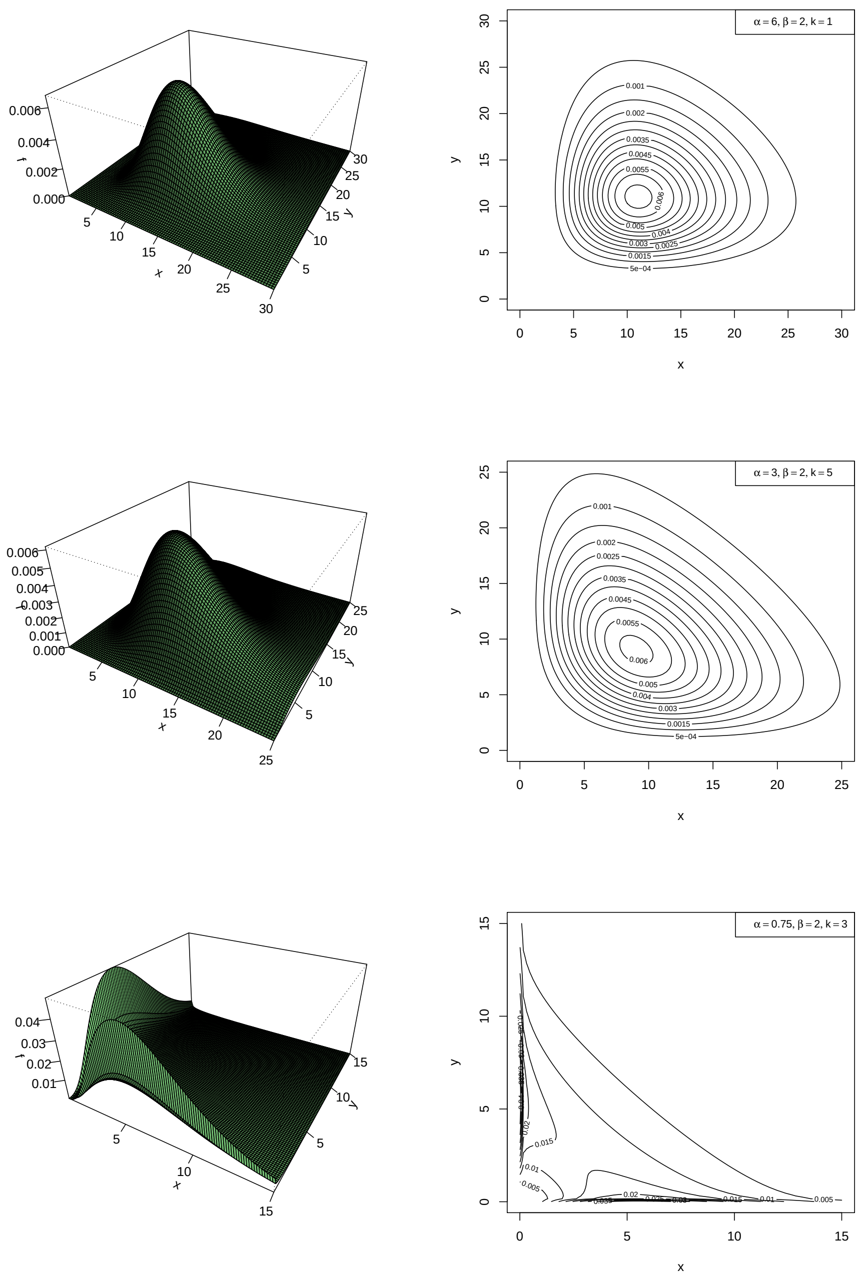

Figure 1. Pdf (1) with contour plots for some selected values of parameters.

in (14) and integrating u, the density $f_{X_1}(x_1)$, in series involving generalized Laguerre polynomials, is derived as

$$f_{X_1}(x_1) = C(\alpha, \beta, k) x_1^{2\alpha+k-1} \exp\left(-\frac{x_1}{\beta}\right) \sum_{j=0}^{\infty} \frac{1}{\alpha + j} L_j^{(\alpha+k)}\left(\frac{x_1}{\beta}\right), \quad x_1 > 0. \tag{15}$$

where $L_j^{(\alpha)}(\cdot)$ is the generalized Laguerre polynomial (see Appendix for the definition).

From the joint pdf (1) and the marginal density of X given in (12), the conditional pdf of X_2 given $X_1 = x_1$ is given by

$$f(x_2 \mid x_1) \quad = \quad \frac{x_2^{\alpha-1}(x_1+x_2)^k \exp(-x_2/\beta)}{\sum_{i=0}^{k}\binom{k}{i}\beta^{\alpha+i}\Gamma\left(\alpha+i\right)x_1^{k-i}}. \tag{16}$$

Also, the conditional pdf of X_1 given $X_2 = x_2$ is given by

$$f(x_1 \mid x_2) = \frac{x_1^{\alpha-1}(x_1+x_2)^k \exp(-x_1/\beta)}{\sum_{i=0}^{k}\binom{k}{i}\beta^{\alpha+i}\Gamma\left(\alpha+i\right)x_2^{k-i}}. \tag{17}$$

3. Moments

By definition

$$E(X_1^m X_2^n) = C(\alpha,\beta,k)\int_0^\infty\int_0^\infty x_1^m x_2^n (x_1x_2)^{\alpha-1}(x_1+x_2)^k \exp\left[-\frac{1}{\beta}(x_1+x_2)\right]dx_1dx_2.$$

Substituting $s = x_1 + x_2$ and $r = x_1/(x_1 + x_2)$ with the Jacobian $J(x_1, x_2 \to r, s) = s$ in the above integral, one gets

$$\begin{aligned} E(X_1^m X_2^n) &= C(\alpha,\beta,k)\int_0^1 r^{\alpha+m-1}(1-r)^{\alpha+n-1}dr\int_0^\infty s^{2\alpha+m+n+k-1}\exp\left(-\frac{s}{\beta}\right)ds \\ &= C(\alpha,\beta,k)\frac{\Gamma(\alpha+m)\Gamma(\alpha+n)}{\Gamma(2\alpha+m+n)}\beta^{2\alpha+m+n+k}\Gamma(2\alpha+m+n+k), \end{aligned}$$

where the last line has been obtained by using beta and gamma integrals. Finally, simplifying the above expression, we get

$$E(X_1^m X_2^n) = \beta^{m+n}\frac{\Gamma(\alpha+m)\Gamma(\alpha+n)(2\alpha+m+n)_k}{\Gamma^2(\alpha)(2\alpha)_k}.$$

Further, substituting appropriately in the above expression, one gets

$$E[(X_1X_2)^h] = \beta^{2h}\frac{\Gamma^2(\alpha+h)(2\alpha+2h)_k}{\Gamma^2(\alpha)(2\alpha)_k},$$

$$E(X_1X_2) = \frac{\beta^2\alpha(2\alpha+k)(2\alpha+k+1)}{2(2\alpha+1)},$$

$$E(X_1^2X_2) = \frac{\beta^3\alpha(2\alpha+k)(2\alpha+k+1)(2\alpha+k+2)}{4(2\alpha+1)},$$

$$E(X_1^3X_2) = \frac{\beta^4\alpha(\alpha+2)(2\alpha+k)(2\alpha+k+1)(2\alpha+k+2)(2\alpha+k+3)}{4(2\alpha+1)(2\alpha+3)}$$

$$E(X_1^2X_2^2) = \frac{\beta^4\alpha(\alpha+1)(2\alpha+k)(2\alpha+k+1)(2\alpha+k+2)(2\alpha+k+3)}{4(2\alpha+1)(2\alpha+3)}$$

$$E(X_i) = \frac{\beta(2\alpha+k)}{2}, \quad i = 1, 2,$$

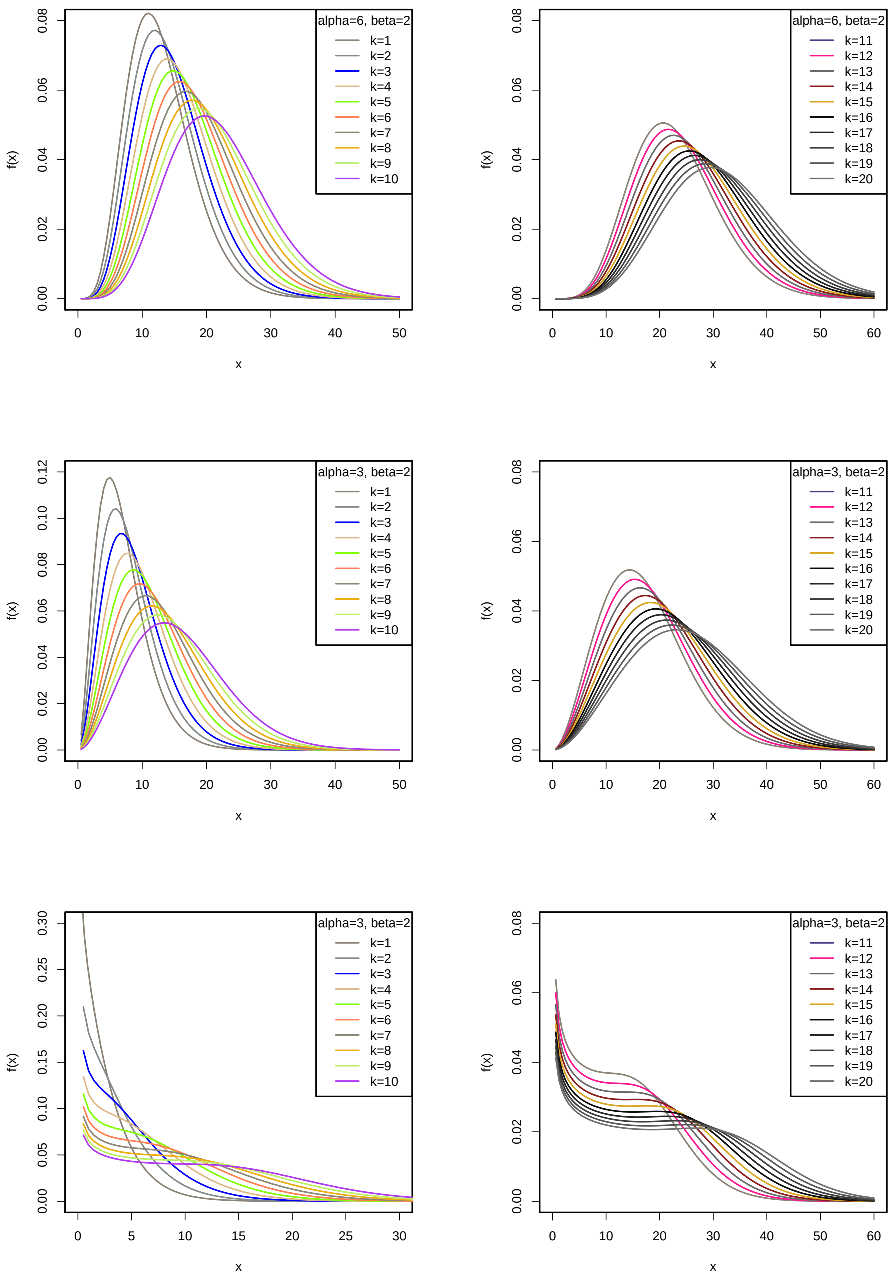

Figure 2. Plots of pdf (12) for some selected values of parameters.

$$E(X_i^2) = \frac{\beta^2(\alpha+1)(2\alpha+k)(2\alpha+k+1)}{2(2\alpha+1)}, \quad i=1,2,$$

$$E(X_i^3) = \frac{\beta^3(\alpha+2)(2\alpha+k)(2\alpha+k+1)(2\alpha+k+2)}{4\,(2\alpha+1)}, \quad i=1,2,$$

and

$$E(X_i^4) = \frac{\beta^4(\alpha+2)(\alpha+3)(2\alpha+k)(2\alpha+k+1)(2\alpha+k+2)(2\alpha+k+3)}{4(2\alpha+1)(2\alpha+3)}.$$

Further, variances, covariances, correlation and several higher central moments are derived as

$$\mu_{11} = -\frac{k\beta^2(2\alpha+k)}{4(2\alpha+1)},$$

$$\mu_{20} = \mu_{02} = \frac{\beta^2(2\alpha+k)(4\alpha+k+2)}{4(2\alpha+1)}, \quad i = 1, 2,$$

$$\mu_{30} = \mu_{03} = \frac{\beta^3(2\alpha+k)(8\alpha+3k+4)}{4(2\alpha+1)},$$

$$\operatorname{corr}(X_1, X_2) = -\frac{k}{4\alpha+k+2},$$

$$\beta_{i1} = \sqrt{\frac{4(2\alpha+1)(8\alpha+3k+4)^2}{(2\alpha+k)(4\alpha+k+2)^3}}, \quad i = 1, 2,$$

$$\mu_{21} = -\frac{\beta^3 k(2\alpha+k)}{4(2\alpha+1)},$$

$$\mu_{31} = -\frac{3\beta^4 k(2\alpha+k)(2\alpha+k+2)(4\alpha+k+4)}{16(2\alpha+1)(2\alpha+3)},$$

$$\mu_{22} = \frac{(k+2\alpha)\left[3k^3+2k^2(6+7\alpha)+4k\alpha(11+8\alpha)+8\alpha(2\alpha+1)(2\alpha+3)\right]\beta^4}{16(2\alpha+1)(2\alpha+3)},$$

where

$$\mu_{ij} = E[(X_1-\mu)^i(X_2-\mu)^j].$$

4. Moment Generating Function

By definition, the joint mgf of X_1 and X_2 is given by

$$M_{X_1,X_2}(t_1,t_2) = C(\alpha,\beta,k)\int_0^\infty\int_0^\infty (x_1x_2)^{\alpha-1}(x_1+x_2)^k \exp\left[t_1x_1+t_2x_2-\frac{1}{\beta}(x_1+x_2)\right]dx_1\,dx_2. \tag{18}$$

Substituting $x_1 = rs$ and $x_2 = s(1-r)$ in (18) with the Jacobian $J(x_1, x_2 \to r, s) = s$ and integrating s, we get

$$M_{X_1,X_2}(t_1,t_2) = C(\alpha,\beta,k)\beta^{2\alpha+k}\Gamma(2\alpha+k) \times \int_0^1 [r(1-r)]^{\alpha-1}[r(1-t_1\beta)+(1-r)(1-t_2\beta)]^{-(2\alpha+k)}dr, \tag{19}$$

where $1-t_1\beta > 0$ and $1-t_2\beta > 0$. Now, writing

$$[r(1-t_1\beta)+(1-r)(1-t_2\beta)]^{-(2\alpha+k)} = (1-t_2\beta)^{-(2\alpha+k)}\left[1-r\left(1-\frac{1-t_1\beta}{1-t_2\beta}\right)\right]^{-(2\alpha+k)}, \quad \frac{1-t_1\beta}{1-t_2\beta} < 1,$$

in (19) and integrating r, we get

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= C(\alpha,\beta,k)\beta^{2\alpha+k}(1-t_2\beta)^{-(2\alpha+k)}\Gamma(2\alpha+k) \\ &\quad\times \int_0^1 [r(1-r)]^{\alpha-1}\left[1-r\left(1-\frac{1-t_1\beta}{1-t_2\beta}\right)\right]^{-(2\alpha+k)} dr \\ &= C(\alpha,\beta,k)\beta^{2\alpha+k}(1-t_2\beta)^{-(2\alpha+k)}\Gamma(2\alpha+k) \\ &\quad\times \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} F\left(\alpha, 2\alpha+k; 2\alpha; 1-\frac{1-t_1\beta}{1-t_2\beta}\right), \end{aligned} \tag{20}$$

where the last line has been obtained by using the integral representation of the Gauss hypergeometric function given in (A.1). Finally, substituting for $C(\alpha,\beta,k)$ and simplifying, we get

$$M_{X_1,X_2}(t_1,t_2) = (1-t_2\beta)^{-(2\alpha+k)} F\left(\alpha, 2\alpha+k; 2\alpha; 1-\frac{1-t_1\beta}{1-t_2\beta}\right).$$

For $t_1 = t_2 = t$, we have

$$M_{X_1,X_2}(t,t) = M_{X_1+X_2}(t) = (1-t\beta)^{-(2\alpha+k)}$$

which is the mgf of a gamma random variable with shape parameter $2\alpha+k$ and scale parameter β.

5. Entropies

In this section, exact forms of Rényi and Shannon entropies are derived for the bivariate gamma distribution defined in this article.

Let $(\mathcal{X},\mathcal{B},\mathcal{P})$ be a probability space. Consider a pdf f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [24]. It is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x)\log f(x)\,\mathrm{d}\mu. \tag{21}$$

One of the main extensions of the Shannon entropy was defined by Rényi [21]. This generalized entropy measure is given by

$$H_R(\eta,f) = \frac{\log G(\eta)}{1-\eta} \qquad (\text{for } \eta>0 \text{ and } \eta\neq 1), \tag{22}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^\eta d\mu.$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η, while Shannon entropy (21) is obtained from (22) for $\eta\uparrow 1$. For details see Nadarajah and Zografos [17], Zografos and Nadarajah [32] and Zografos [31].

Theorem 5.1
For the bivariate gamma distribution defined by the pdf (1), the Rényi and the Shannon entropies are given by

$$\begin{aligned} H_R(\eta,f) = \frac{1}{1-\eta}\Big[&\eta\ln C(\alpha,\beta,k) + [\eta(2\alpha+k-2)+2]\ln\left(\frac{\beta}{\eta}\right) \\ &+ 2\ln\Gamma\left[\eta(\alpha-1)+1\right] + \ln\Gamma[\eta(2\alpha+k-2)+2] - \ln\Gamma[\eta(2\alpha-2)+2]\Big] \end{aligned}$$

and

$$H_{SH}(f) = -\ln C(\alpha,\beta,k) - [(2\alpha+k-2)\ln\beta - (2\alpha+k) + 2(\alpha-1)\psi(\alpha) + (2\alpha+k-2)\psi(2\alpha+k) - (2\alpha-2)\psi(2\alpha)],$$

respectively, where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function.

Proof
For $\eta > 0$ and $\eta \neq 1$, using the joint density of X_1 and X_2 given by (1), we have

$$\begin{aligned} G(\eta) &= \int_0^\infty \int_0^\infty f^\eta(x_1,x_2;\alpha,\beta,k)\,dx_2\,dx_1 \\ &= [C(\alpha,\beta,k)]^\eta \int_0^\infty \int_0^\infty (x_1x_2)^{\eta(\alpha-1)}\,(x_1+x_2)^{\eta k} \exp\left[-\frac{\eta}{\beta}(x_1+x_2)\right] dx_2\,dx_1 \\ &= [C(\alpha,\beta,k)]^\eta \int_0^\infty \int_0^1 [r(1-r)]^{\eta(\alpha-1)} s^{\eta(2\alpha-2)+\eta k+1} \exp\left(-\frac{\eta}{\beta}s\right) dr\,ds, \end{aligned}$$

where the last line has been obtained by substituting $s = x_1 + x_2$ and $r = x_1/(x_1 + x_2)$. Finally, evaluating above integrals by using gamma and beta integrals and simplifying the resulting expression, we get

$$G(\eta) = [C(\alpha,\beta,k)]^\eta \frac{\Gamma^2[\eta(\alpha-1)+1]\Gamma[\eta(2\alpha+k-2)+2]}{\Gamma[\eta(2\alpha-2)+2]} \left(\frac{\beta}{\eta}\right)^{\eta(2\alpha+k-2)+2}.$$

Now, taking logarithm of $G(\eta)$ and using (22) we get $H_R(\eta, f)$. The Shannon entropy is obtained from $H_R(\eta, f)$ by taking $\eta \uparrow 1$ and using L'Hopital's rule. □

6. Sum, Quotient and Product

In this section we derive the distributions of $X_1 + X_2$, $X_1/(X_1 + X_2)$, X_1X_2, and X_1/X_2 when X_1 and X_2 follow a bivariate gamma distribution defined in (1).

Theorem 6.1
Let $(X_1, X_2) \sim \text{BGa}(\alpha, \beta, k)$, and define $R = X_1/(X_1 + X_2)$ and $S = X_1 + X_2$. Then, R and S are independent, the distribution of R is beta with both the parameters α and the distribution of S is gamma with shape parameter $2\alpha + k$ and scale parameter β.

Proof
Substituting $x_1 = rs$ and $x_2 = s(1 - r)$ with the Jacobian $J(x_1, x_2 \to r, s) = s$, in the joint density of X_1 and X_2, we obtain the joint density of R and S as

$$C(\alpha,\beta,k)[r(1-r)]^{\alpha-1} s^{2\alpha+k-1} \exp\left(-\frac{s}{\beta}\right), \tag{23}$$

where $0 < r < 1$ and $s > 0$. Now, from (23), the desired result is obtained. □

Corollary 6.1.1
Both X_1/X_2 and X_2/X_1 have inverted beta distribution with parameters α and α.

Theorem 6.2
Let $(X_1, X_2) \sim \text{BGa}(\alpha, \beta, k)$, and define $P = X_1X_2$. Then, the density of P is given by

$$C(\alpha,\beta,k)2p^{\alpha+k/2-1} \sum_{j=0}^{k} \binom{k}{j} K_{k-2j}\left(2\frac{\sqrt{p}}{\beta}\right), \quad p > 0.$$

Proof
Transforming $X_1 = X$ and $P = X_1 X_2$ with the Jacobian $J(x_1, x_2 \to p) = 1/x$ in the joint density of X_1 and X_2 and integrating x, we obtain the density of P as

$$
\begin{aligned}
& C(\alpha,\beta,k)p^{\alpha-1}\int_0^\infty \frac{1}{x}\left(x+\frac{p}{x}\right)^k \exp\left[-\frac{1}{\beta}\left(x+\frac{p}{x}\right)\right]dx \\
&= C(\alpha,\beta,k)p^{\alpha-1}\sum_{j=0}^{k}\binom{k}{j}p^j\int_0^\infty x^{k-2j-1}\exp\left[-\frac{1}{\beta}\left(x+\frac{p}{x}\right)\right]dx. \qquad (24)
\end{aligned}
$$

Now, using the integral (Gradshteyn and Ryzhik [4, Eq. 3.471.9]),

$$
\int_0^\infty \exp\left(-az-\frac{b}{z}\right)z^{\nu-1}\,dz = 2\left(\frac{b}{a}\right)^{\nu/2}K_\nu(2\sqrt{ab}), \quad a>0. \quad b>0,
$$

where K_ν is the modified Bessel function of the second kind, we obtain the desired result.

□

Next two theorems deal with bivariate distributions of $(X_1/Y, X_2/Y)$ and $(X_1/U, X_2/U)$, where $(X_1, X_2) \sim \text{BGa}(\alpha,\beta,k)$, $Y \sim \text{Ga}(\nu,\beta)$ and $U \sim B(a,b)$.

Theorem 6.3
Let $(X_1, X_2) \sim \text{BGa}(\alpha,\beta,k)$, and $Y \sim \text{Ga}(\nu,\beta)$ be independent. Then, the joint density of $Z_1 = X_1/Y$ and $Z_2 = X_2/Y$ is given by

$$
\frac{\Gamma(2\alpha)\Gamma(2\alpha+k+\nu)}{\Gamma^2(\alpha)\Gamma\left(2\alpha+k\right)\Gamma(\nu)}\frac{(z_1z_2)^{\alpha-1}(z_1+z_2)^k}{(1+z_1+z_2)^{2\alpha+k+\nu}}, \quad z_1>0, \quad z_2>0.
$$

Proof
Transforming $X_1 = Z_1Y$ and $X_2 = Z_2Y$ with the Jacobian $J(x_1, x_2, \to z_1, z_2) = y^2$ in the join density of (X_1, X_2) and Y, the joint density of (Z_1, Z_2) and Y is obtained as

$$
\frac{C(\alpha,\beta,k)}{\Gamma(\nu)\beta^\nu}(z_1z_2)^{\alpha-1}(z_1+z_2)^k y^{2\alpha+k+\nu-1}\exp\left[-\frac{(1+z_1+z_2)y}{\beta}\right], \quad z_1>0, \quad z_2>0, \quad y>0.
$$

Now, integrating y by using gamma integral, we get the desired result. □

For $k = 0$, the variables X_1, X_2 and Y are independent gamma with scale parameter β and therefore $(X_1/Y, X_2/Y)$ has a Dirichlet type 2 distribution.

Theorem 6.4
Let $(X_1, X_2) \sim \text{BGa}(\alpha,\beta,k)$, and $U \sim B(a,b)$ be independent. Then, the joint density of $Z_1 = X_1/U$ and $Z_2 = X_2/U$ is given by

$$
\begin{aligned}
& \frac{\beta^{-(2\alpha+k)}\Gamma(2\alpha)\Gamma(2\alpha+k+a)\Gamma(a+b)}{\Gamma^2(\alpha)\Gamma(2\alpha+k)\Gamma(2\alpha+k+a+b)\Gamma(a)}(z_1z_2)^{\alpha-1}(z_1+z_2)^k \\
& \times\Phi\left(2\alpha+k+a; 2\alpha+k+a+b; -\frac{z_1+z_2}{\beta}\right), \quad z_1>0, \quad z_2>0.
\end{aligned}
$$

Proof
Transforming $X_1 = Z_1U$ and $X_2 = Z_2U$ with the Jacobian $J(x_1, x_2, \to z_1, z_2) = u^2$ in the join density of (X_1, X_2) and U, the joint density of (Z_1, Z_2) and U is obtained as

$$
\frac{C(\alpha,\beta,k)}{B(a,b)}(z_1z_2)^{\alpha-1}(z_1+z_2)^k u^{2\alpha+k+a-1}(1-u)^{b-1}\exp\left[-\frac{(z_1+z_2)u}{\beta}\right],
$$

where $z_1>0$, $z_2>0$, and $0<u<1$. Now, integrating u by using integral representation of confluent hypergeometric function given in (A.2), we get the desired result. □

7. Estimation

Let $(X_{11}, X_{12}), \cdots, (X_{n1}, X_{n2})$ be a random sample from BGa(α, β, k). The log-likelihood function, denoted by $l(\alpha, \beta)$, is given by

$$\begin{aligned} l(\alpha, \beta) &= n\left[\ln\Gamma(2\alpha) - (2\alpha + k)\ln\beta - 2\ln\Gamma(\alpha) - \ln\Gamma(2\alpha + k)\right] \\ &\quad + (\alpha - 1)\sum_{i=1}^{n}(\ln x_{i1} + \ln x_{i2}) + k\sum_{i=1}^{n}\ln(x_{i1} + x_{i2}) - \frac{1}{\beta}\sum_{i=1}^{n}(x_{i1} + x_{i2}). \end{aligned}$$

Now, differentiating $l(\alpha, \beta)$ w.r.t. α, we get

$$\frac{\partial l(\alpha, \beta)}{\partial \alpha} = n\left[2\psi(2\alpha) - 2\ln\beta - 2\psi(\alpha) - 2\psi(2\alpha + k)\right] + \sum_{i=1}^{n}(\ln x_{i1} + \ln x_{i2}).$$

Using the duplication formula for digamma function, namely,

$$2\psi(2z) = \ln 4 + \psi(z) + \psi\left(z + \frac{1}{2}\right)$$

we obtain

$$\frac{\partial l(\alpha, \beta)}{\partial \alpha} = n\left[\ln 4 + \psi\left(\alpha + \frac{1}{2}\right) - \psi(\alpha) - 2\ln\beta - 2\psi(2\alpha + k)\right] + \sum_{i=1}^{n}(\ln x_{i1} + \ln x_{i2}).$$

Further,

$$\frac{\partial l(\alpha, \beta)}{\partial \beta} = -\frac{n(2\alpha + k)}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_{i1} + x_{i2}),$$

$$\frac{\partial l(\alpha, \beta)}{\partial \alpha \partial \beta} = -\frac{2n}{\beta},$$

$$\frac{\partial^2 l(\alpha, \beta)}{\partial \alpha^2} = n\psi_1\left(\alpha + \frac{1}{2}\right) - n\psi_1(\alpha) - 4n\psi_1(2\alpha + k),$$

$$\frac{\partial^2 l(\alpha, \beta)}{\partial \beta^2} = \frac{n(2\alpha + k)}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n}(x_{i1} + x_{i2}),$$

$$E\left[\frac{\partial l(\alpha, \beta)}{\partial \alpha \partial \beta}\right] = -\frac{2n}{\beta},$$

$$E\left[\frac{\partial^2 l(\alpha, \beta)}{\partial \alpha^2}\right] = n\psi_1\left(\alpha + \frac{1}{2}\right) - n\psi_1(\alpha) - 4n\psi_1(2\alpha + k),$$

$$E\left[\frac{\partial^2 l(\alpha, \beta)}{\partial \beta^2}\right] = -\frac{n(2\alpha + k)}{\beta^2}.$$

For a given observation vector (x_1, x_2), the Fisher information matrix for the bivariate distribution given by the density (1) is defined as

$$\begin{pmatrix} -\psi_1(\alpha+1/2)+\psi_1(\alpha)+4\psi_1(2\alpha+k) & 2/\beta \\ 2/\beta & (2\alpha+k)/\beta^2 \end{pmatrix}.$$

Further

$$\frac{\partial l(\alpha,\beta)}{\partial\beta} = -\frac{n(2\alpha+k)}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_{i1}+x_{i2}) = 0$$

gives

$$(2\alpha+k)\beta = \bar{x}_1 + \bar{x}_2 \tag{25}$$

and

$$\frac{\partial l(\alpha,\beta)}{\partial\alpha} = n\left[2\psi(2\alpha) - 2\ln\beta - 2\psi(\alpha) - 2\psi(2\alpha+k)\right] + \sum_{i=1}^{n}(\ln x_{i1} + \ln x_{i2}) = 0$$

gives

$$\psi(2\alpha+k) - \psi(2\alpha) + \ln\beta + \psi(\alpha) = \frac{1}{2}\ln(\tilde{x}_1\tilde{x}_2),$$

where $\tilde{x}_i = \prod_{j=1}^{n} x_{ij}^{1/n}$, $i = 1, 2$. Further, using

$$\psi(z+N) - \psi(z) = \sum_{j=0}^{N-1}\frac{1}{z+j}$$

we have

$$\sum_{j=0}^{k-1}\frac{1}{2\alpha+j} + \ln\beta + \psi(\alpha) = \frac{1}{2}\ln(\tilde{x}_1\tilde{x}_2). \tag{26}$$

Thus, by solving numerically (25) and (26), the MLEs of α and β can be obtained.

8. Simulation

In this section a simulation study is conducted to evaluate the performance of maximum likelihood method. Samples of size $n = 30, 50, 200, 500$ from Equation (1) for selected values of parameters are generated by MCMC methods (Gibbs Metropolise, Markov Chain Monte Carlo Metropolise, Metropolise, Metropolise gaussian, random walk Metropolise and Metropolise-Hastings). For $\alpha = 6$, $\beta = 2$ and $k = 1, 4, 8$ that $\rho = -\frac{1}{27}$, $\rho = -\frac{4}{30}$ and $\rho = -\frac{8}{34}$ respectively, the random walk Metropolis algorithm method has better results. When $\alpha = 0.75$, $\beta = 2$, $k = 1, 4, 8$ that $\rho = -\frac{1}{6}$, $\rho = -\frac{2}{7}$ and $\rho = -\frac{3}{8}$, the Gibbs sampling method provides better results and is used to simulate samples.

For each sample, MLEs for α, β and k based on the numerical procedures are computed. This procedure is repeated five hundred times and $(\widehat{\alpha}, \widehat{\beta}, \widehat{k})$, the bias (Ab) and the mean squared error (MSE) are obtained by using Monte Carlo method. The results are reported in Tables (1) and (2). Figures 3, 4 and 5 show the simulation data and contour plots for $\alpha = 6$, $\beta = 2$ and $k = 1, 4, 8$ with $n = 200$. Figure 6 shows pairs style of random walk Metropolis method for $\alpha = 6$, $\beta = 2$ and $k = 1$ with $n = 500$ and Figure 7 exhibits pairs style of Gibbs sampling method for $\alpha = 0.75$, $\beta = 2$ and $k = 4$ with $n = 200$.

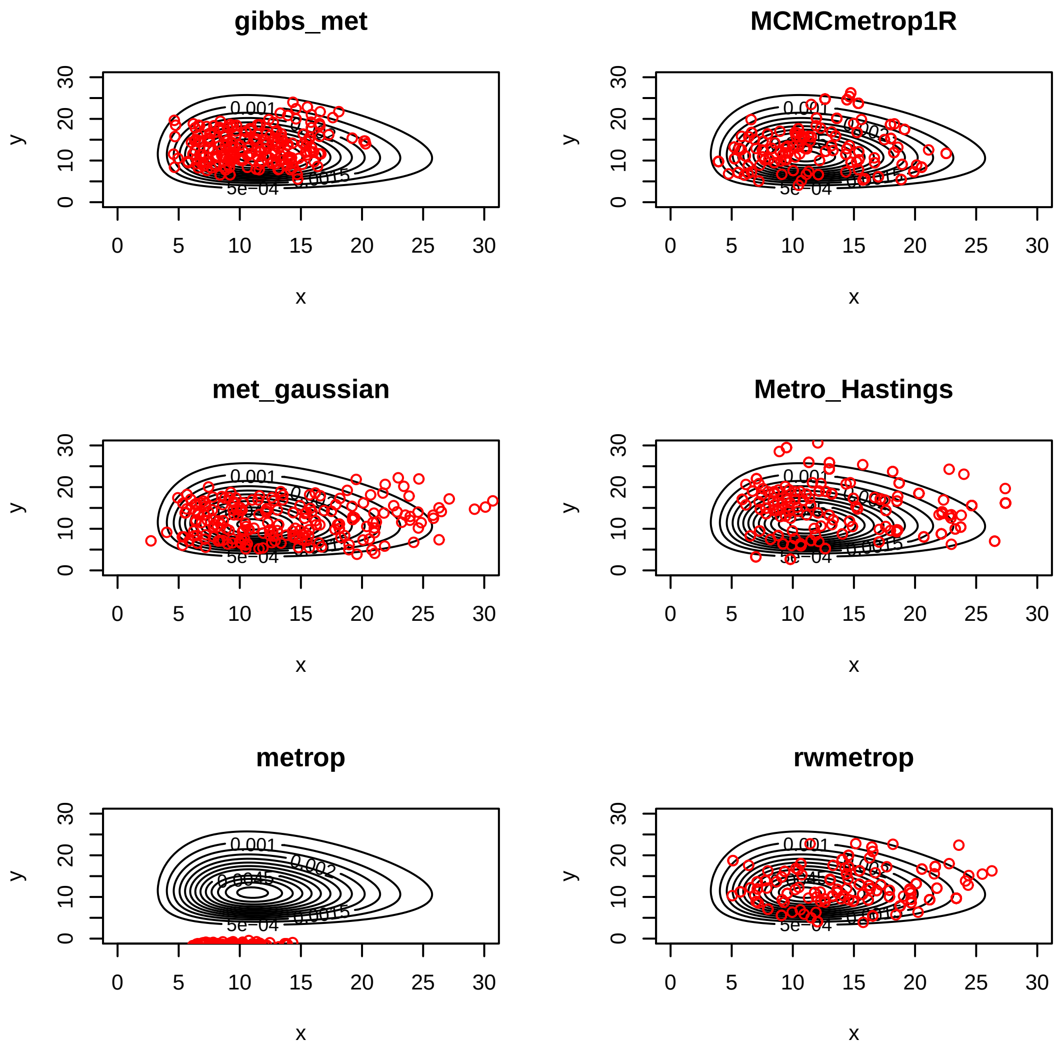

Figure 3. Simulation data and contour plots for different methods for $\alpha = 6$, $\beta = 2$ and $k = 1$ with $n = 200$.

9. Multivariate generalization

The multivariate generalization of (1) can be defined as follows:

$$C_n(\alpha, \beta, k)(x_1 x_2 \cdots x_n)^{\alpha-1}(x_1 + x_2 + \cdots + x_n)^k \exp\left[-\frac{1}{\beta}(x_1 + x_2 + \cdots + x_n)\right], \tag{27}$$

where $x_1 > 0, x_2 > 0, \ldots, x_n > 0$ and $C(\alpha, \beta, k)$ is the normalizing constant given by

$$C_n(\alpha, \beta, k) = \frac{\Gamma(n\alpha)}{\beta^{n\alpha+k}\Gamma^n(\alpha)\Gamma(n\alpha + k)}. \tag{28}$$

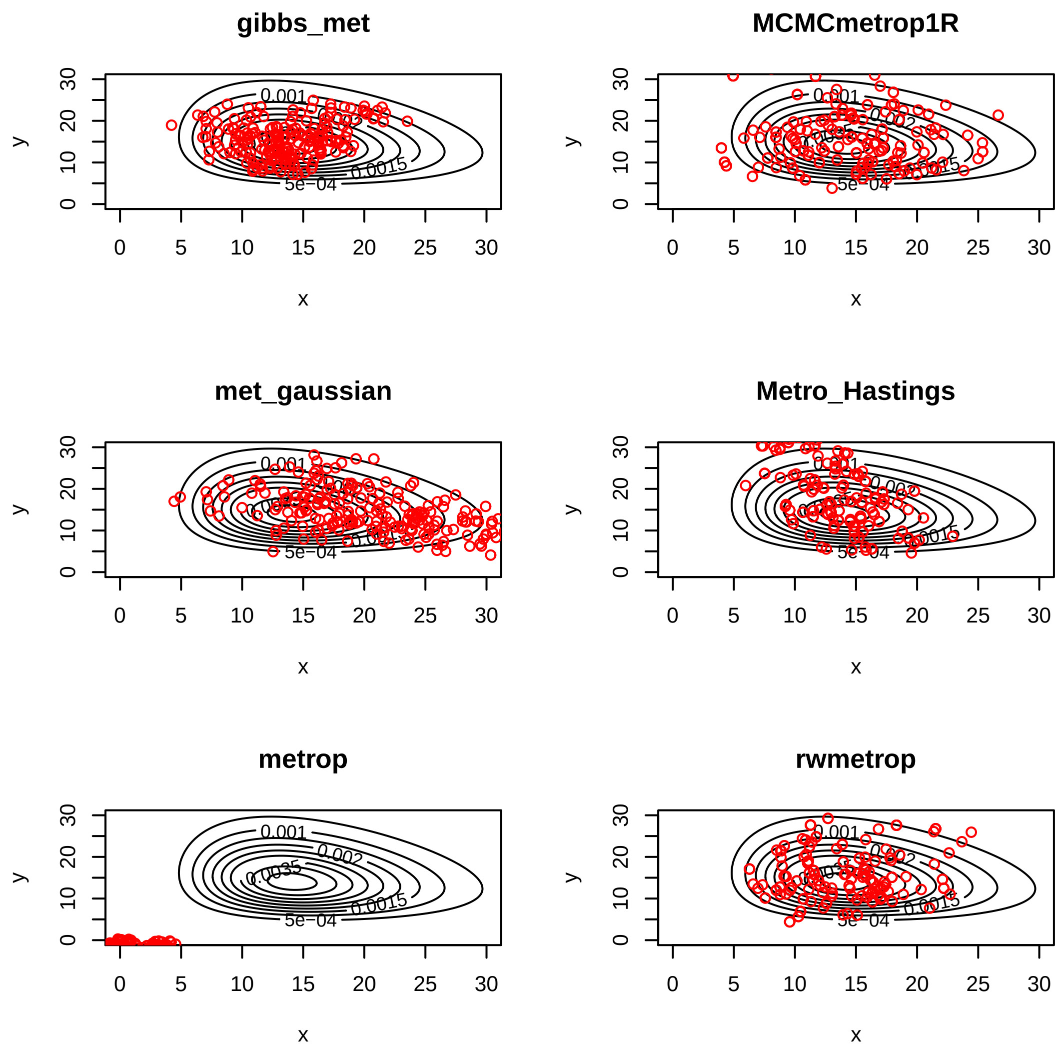

Figure 4. Simulation data and contour plots for different methods for $\alpha = 6$, $\beta = 2$ and $k = 4$ with $n = 200$.

Appendix

The Gauss hypergeometric function, denoted by $F(a, b; c; z)$, and confluent hypergeometric function, denoted by $\Phi(b; c; z)$, for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, are defined as (see Luke [10]),

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \, \mathrm{d}t, \quad |\arg(1-z)| < \pi, \tag{A.1}$$

and

$$\Phi(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp(zt) \, \mathrm{d}t. \tag{A.2}$$

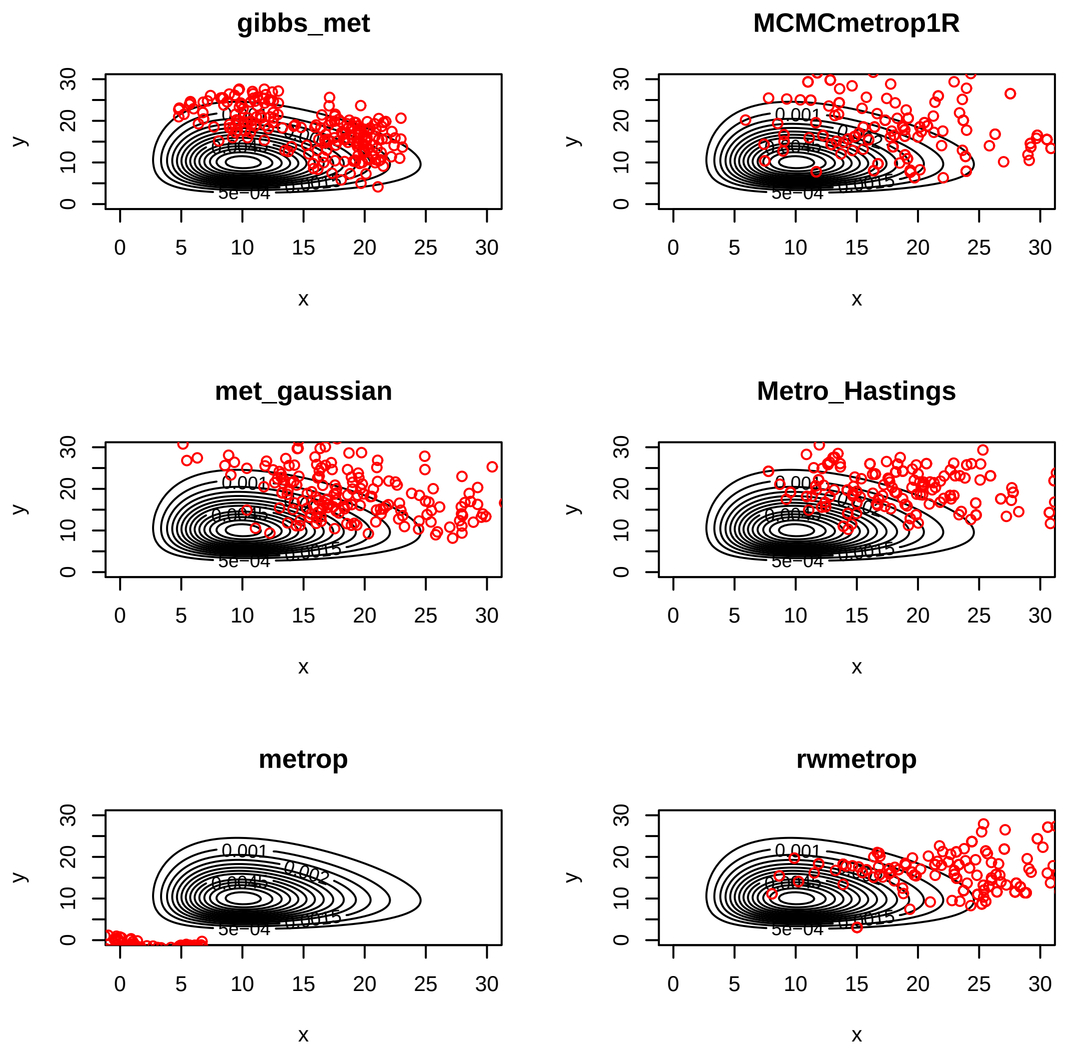

Figure 5. Simulation data and contour plots for different methods for $\alpha = 6$, $\beta = 2$ and $k = 8$ with $n = 200$.

Using the series expansion of $(1 - zt)^{-a}$ in (A.1) and $\exp(zt)$ in (A.2), the following series representations of the hypergeometric functions can be obtained:

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \tag{A.3}$$

and

$$\Phi(b; c; z) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!}, \tag{A.4}$$

where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \dots,$ and $(a)_0 = 1$.

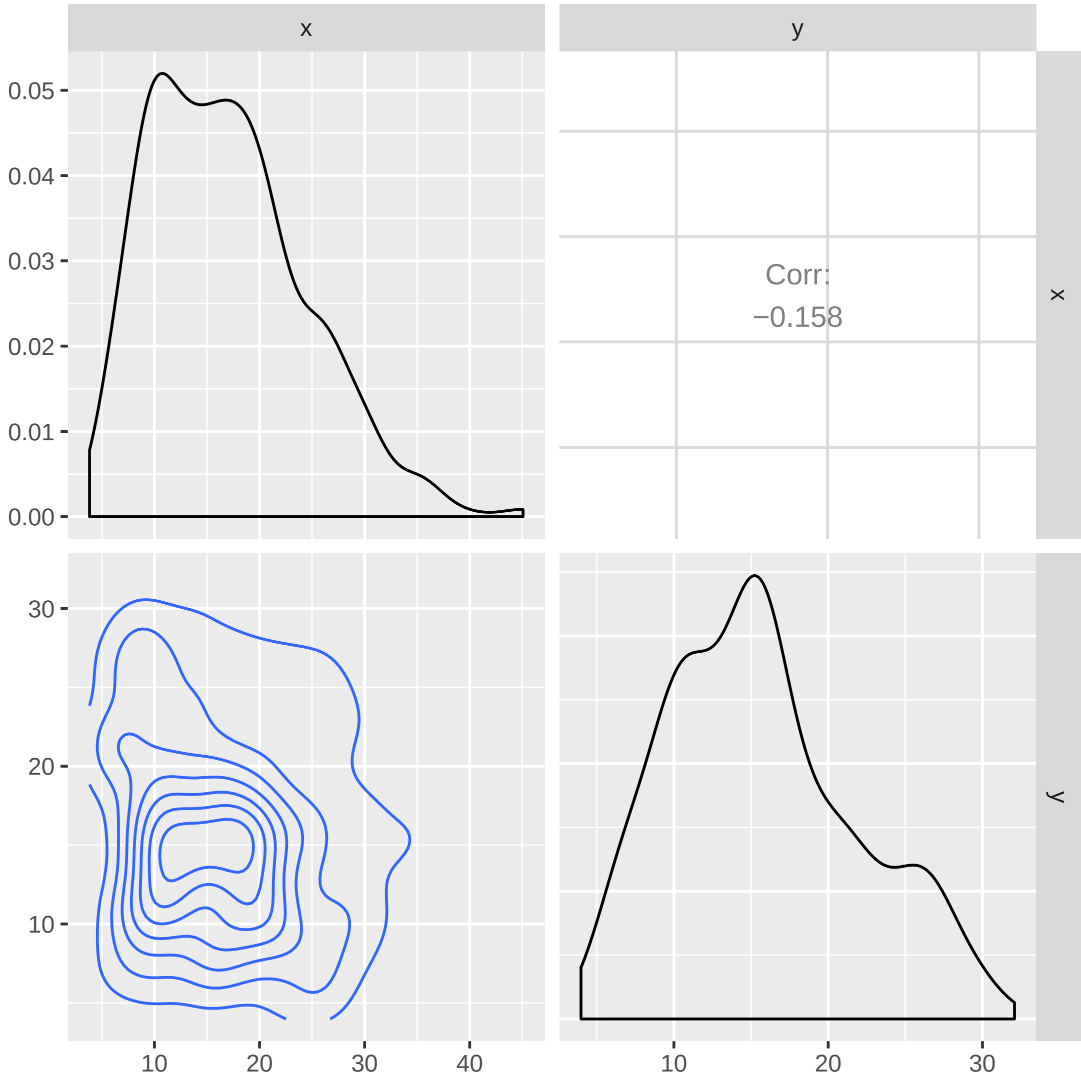

Figure 6. Pairs style of random walk Metropolis method for $\alpha = 6$, $\beta = 2$ and $k = 1$ with $n = 500$.

Also, under suitable conditions, we have (Luke [10, Eq. 3.6(10)]),

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;zy)\,\mathrm{d}z$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}{}_{p+1}F_{q+1}(a_1,\dots,a_p,\alpha;b_1,\dots,b_q,\alpha+\beta;y). \tag{A.5}$$

Lemma A.1
For $a > 0$, $b > 0$ and $k \in \mathbb{N}$, we have

$$\sum_{i=0}^{k}\binom{k}{i}(a)_i\,(b)_{k-i} = (a+b)_k.$$

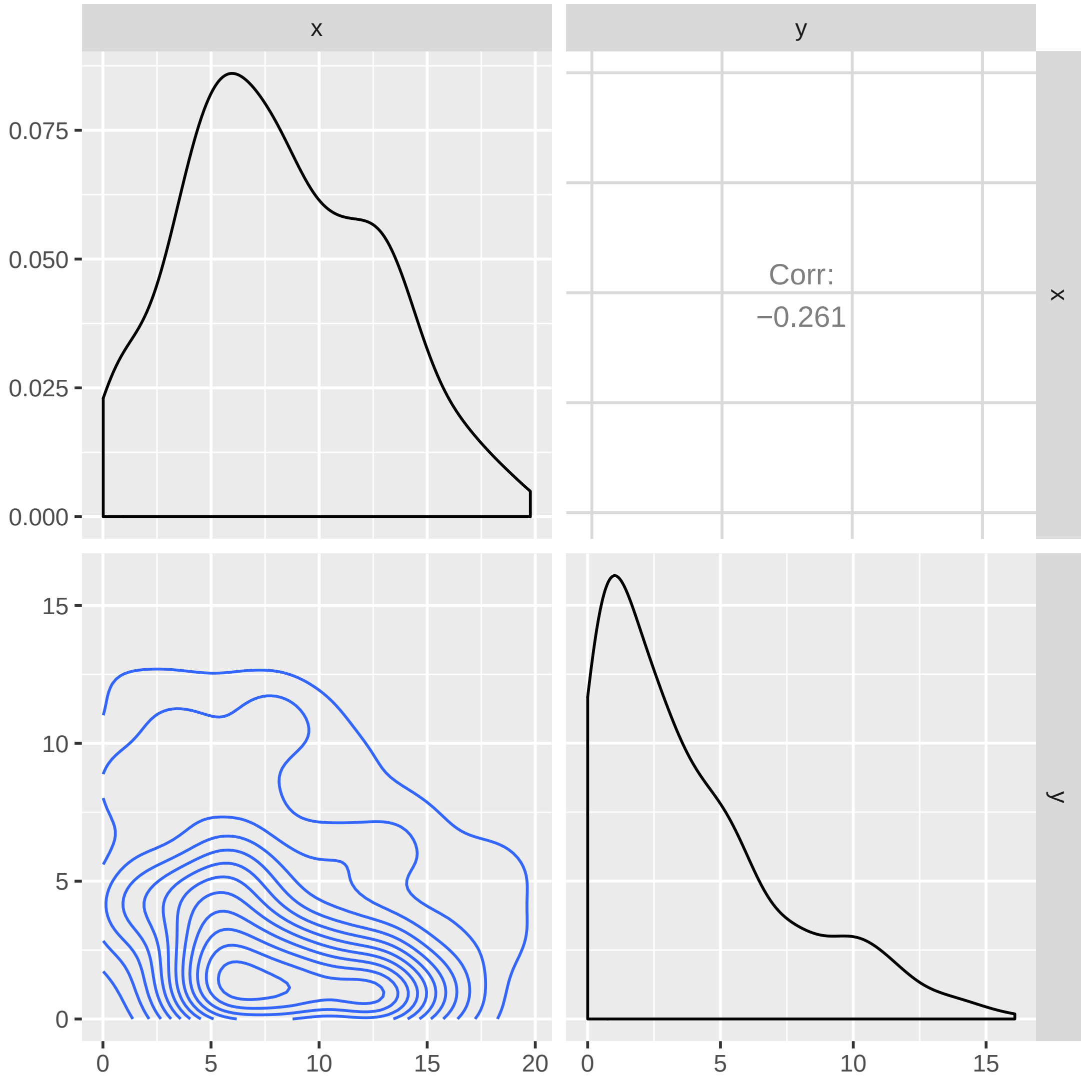

Figure 7. Pairs style of Gibbs sampling method for $\alpha = 0.75$, $\beta = 2$ and $k = 4$ with $n = 200$.

Proof
Writing $(1-\theta)^{-(a+b)}$ as $(1-\theta)^{-a}(1-\theta)^{-b}$ and using power series expansion, for $0<\theta<1$, we get

$$\begin{aligned}(1-\theta)^{-a}(1-\theta)^{-b} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_i(b)_j}{i!\,j!}\theta^{i+j}\\ &= \sum_{k=0}^{\infty}\theta^k\sum_{i+j=k}\frac{(a)_i(b)_j}{i!\,j!}\\ &= \sum_{k=0}^{\infty}\theta^k\sum_{i=0}^{k}\frac{(a)_i(b)_{k-i}}{i!\,(k-i)!}\end{aligned}$$

and

$$(1-\theta)^{-(a+b)} = \sum_{k=0}^{\infty} \frac{(a+b)_k}{k!}\theta^k.$$

Now, comparing the coefficients of θ^k, we get the desired result. □

The generating function of the generalized Laguerre polynomial is

$$(1-t)^{-(a+1)} \exp\left(-\frac{zt}{1-t}\right) = \sum_{j=0}^{\infty} t^j L_j^{(a)}(z).$$

Finally, we define the gamma, beta type 1 and beta type 2 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [7].

Definition A.1
A random variable X is said to have a gamma distribution with parameters $\theta\,(>0), \kappa\,(>0)$, denoted by $X \sim \mathrm{Ga}(\kappa,\theta)$, if its pdf is given by

$$\{\theta^\kappa \Gamma(\kappa)\}^{-1} x^{\kappa-1} \exp\left(-\frac{x}{\theta}\right), \quad x > 0. \tag{A.6}$$

Note that for $\theta = 1$, the above distribution reduces to a standard gamma distribution and in this case we write $X \sim \mathrm{Ga}(\kappa)$.

Definition A.2
A random variable X is said to have a beta type 1 distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by

$$\{B(a,b)\}^{-1} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, \tag{A.7}$$

where $B(a, b)$ is the beta function.

Definition A.3
A random variable X is said to have a beta type 2 (inverted beta) distribution with parameters (a, b), denoted as $X \sim \mathrm{B2}(a, b)$, $a > 0$, $b > 0$, if its pdf is given by

$$\{B(a,b)\}^{-1} x^{a-1}(1+x)^{-(a+b)}, \quad x > 0. \tag{A.8}$$

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Table 1. MLE of simulation with $\alpha = 6$, $\beta = 2$ and $k = 1, 4, 8$

α	β	k	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$Ab(\hat{\alpha})$	$Ab(\hat{\beta})$	$Ab(\hat{k})$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$corr$
6	2	1	30	8.0490032	1.3521385	8.9860000	2.0490032	-0.6478615	7.9860000	26.7930544	1.1103721	0.4534917
			50	7.1348074	1.5613466	5.5640000	1.1348074	-0.4386534	4.5640000	12.2150004	0.7738958	0.4534917
			200	6.3116649	1.8737130	2.2340000	0.3116649	-0.1262870	1.2340000	2.3459757	0.3360376	0.300168
			500	6.12136864	2.01486647	1.11400000	0.12136864	0.01486647	0.11400000	1.00220167	0.18091314	-0.2487341
6	2	4	30	8.5477385	1.3830973	15.0960000	2.5477385	-0.6169027	11.0960000	40.5799509	1.0888138	0.2069484
			50	7.7117254	1.5676554	9.8380000	1.7117254	-0.4323446	5.8380000	19.4659369	0.8266115	-0.3564699
			200	6.6526296	1.7460889	7.0020000	0.6526296	-0.2539111	3.0020000	6.3696766	0.4541197	0.05095428
			500	6.14778027	1.95136312	4.52800000	0.14778027	-0.04863688	0.52800000	0.89259685	0.12620644	0.2078054
6	2	8	30	8.6204338	1.3615853	21.9680000	2.6204338	-0.6384147	13.9680000	33.5312927	1.0200545	-0.4879986
			50	6.16577221	1.94406686	4.68200000	0.16577221	-0.05593314	0.68200000	1.26158982	0.15624082	-0.09814335
			200	6.49986645	1.87237296	9.71000000	0.49986645	-0.12762704	1.71000000	3.06331038	0.25328489	0.1303431
			500	6.23188687	1.96264484	8.60400000	0.23188687	-0.03735516	0.60400000	1.12598240	0.14648568	-0.3217912

Table 2. MLE of simulation with $\alpha = 0.75$, $\beta = 2$ and $k = 1, 4, 8$

α	β	k	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$Ab(\hat{\alpha})$	$Ab(\hat{\beta})$	$Ab(\hat{k})$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$corr$
0.75	2	1	30	0.8503846	1.3038761	3.1420000	0.1003846	-0.6961239	2.1420000	0.1559368	0.9908157	-0.8786252
			50	0.79555454	1.54565705	1.95400000	0.04555454	-0.45434295	0.95400000	0.05917405	0.72396201	-0.09172372
			200	0.76396678	1.91391820	1.21000000	0.01396678	-0.08608180	0.21000000	0.01574066	0.39915125	-0.3923851
			500	0.754664660	1.960157607	1.096000000	0.004664660	-0.039842393	0.096000000	0.005574255	0.162469098	-0.3316668
0.75	2	4	30	1.08576145	0.99054874	14.64000000	0.33576145	-1.00945126	10.64000000	1.29107613	1.46295908	-0.6355327
			50	0.93929384	1.25268403	9.74800000	0.18929384	-0.74731597	5.74800000	0.36158134	1.03915630	0.2254778
			200	0.78231536	1.76455788	5.21800000	0.03231536	-0.23544212	1.21800000	0.04112525	0.45339322	-0.1839379
			500	0.757407912	1.909911596	4.502000000	0.007407912	-0.090088404	0.502000000	0.014233611	0.227038802	-0.4332585
0.75	2	8	30	0.757407912	1.909911596	4.502000000	0.007407912	-0.090088404	0.502000000	0.014233611	0.227038802	-0.4892968
			50	0.90523165	1.55580007	12.70000000	0.15523165	-0.44419993	4.70000000	0.25281668	0.76774585	0.3654835
			200	0.78490199	1.86441825	9.21000000	0.03490199	-0.13558175	1.21000000	0.02967984	0.29849273	-0.5264573
			500	0.76590872	1.93814852	8.55600000	0.01590872	-0.06185148	0.55600000	0.01145940	0.14910234	-0.4487656

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